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Faraday waves: rolls versus squares

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Weakly nonlinear capillary–gravity waves of frequency ω and wavenumber k that are induced on the surface of a liquid in a square cylinder that is subjected to the vertical displacement $a_0 \cos 2\omega t$ are studied on the assumptions that: $0 < \delta < ka_0$, where δ is the linear damping ratio; the dominant modes are $\cos kx$ and $\cos ky$, where x and y are Cartesian coordinates in a horizontal plane. The formulation extends those of Simonelli & Gollub (1989), Feng & Sethna (1989), Nagata (1989, 1991) and Umeki (1991) by incorporating capillarity, cubic forcing and cubic damping. The results are also applicable to a laterally unbounded fluid, but the basic symmetry then is hypothetical rather than imposed by the boundaries. Canonical evolution equations for the modal amplitudes are determined from an average Lagrangian. The fixed points of the evolution equations comprise: (i) the null solution; (ii) an orthogonal pair of *rolls* described by either $\cos kx$ or $\cos ky$; (iii) an orthogonal pair of *squares* described by either $\cos kx + \cos ky$ or $\cos kx - \cos ky$; (iv) coupled-mode solutions for which both modes are active and neither in phase nor in antiphase. The solutions for squares are isomorphic to those for rolls through a linear transformation of the coefficients in the Hamiltonian. The fixed points for rolls and squares lie on separate loci in an energy–frequency plane that intersect the null solution at a pair of pitchfork bifurcations, one of which is definitely supercritical and the other of which may be either subcritical or supercritical. The parametric domain of the various solutions includes subdomains in which squares/rolls are stable/unstable and conversely. In the limiting case of deep-water capillary waves in the threshold domain $0 < ka_0 - \delta < 8\delta^3/9$ all of the rolls and coupled-mode solutions are unstable, while squares are stable except for fixed points between the subcritical bifurcation (if it exists) and the corresponding turning point.

1. Introduction

I consider here weakly nonlinear, parametrically excited, standing waves (Faraday waves) in either a square cylinder of side b or a laterally unbounded (in the sense that the effects of lateral boundaries are negligible) body (*slab*) of water of ambient depth d on a horizontal lower boundary that is subjected to the vertical displacement

$$z_0 = a_0 \cos 2\omega t. \tag{1.1}$$

The essential parameters, which measure depth, drive, damping, detuning (from linear resonance) and capillarity, are kd,

$$\epsilon \equiv ka_0 \tanh kd \ll 1, \quad \alpha \equiv \frac{\delta}{\epsilon} < 1, \quad \beta \equiv \frac{\omega - \omega_k}{\epsilon\omega} = O(1), \quad \sigma \equiv \frac{k^2 l_*^2}{1+k^2 l_*^2}, \tag{1.2a–d}$$

where k is the basic wavenumber, δ is the linear damping ratio of the wave,

$$\omega_k^2 = (gk + \hat{T}k^3)\tanh kd \equiv gk(1 + k^2 l_*^2)\tanh kd, \tag{1.3}$$

$\hat{T}$ is the kinematic surface tension, and $l_* \equiv (\hat{T}/g)^{\frac{1}{2}}$ is the capillary length. The lateral boundary condition determines $k = \pi/b$, and hence also ω_k, for the square cylinder, but k is determined *a priori* for the slab only within $1 + O(\epsilon)$ through the restriction $\beta = O(1)$. I follow Simonelli & Gollub (1989), Feng & Sethna (1989), Nagata (1989, 1991) and Umeki (1991) in focusing on a degenerate pair of modes that, by virtue of the square symmetry, differ only by a rotation of $\frac{1}{2}\pi$† and extend their formulations by incorporating capillarity, cubic damping and cubic forcing.

Faraday waves are reviewed by Miles & Henderson (1990). They appear at a threshold for which $\epsilon = \delta$ (linear excitation balances linear damping) and $\beta = 0$ (linear inertial forces balance linear restoring forces) independently of mode shape. If $\delta < \epsilon \ll 1$ and $\beta = O(1)$ a stationary wave of $O(a_0/\epsilon^{\frac{1}{3}})$ amplitude, for which cubical inertial and capillary forces balance the third-order imbalance among the linear forces, is possible. This balance constrains the frequency and amplitude to lie along a certain resonance curve and therefore determines both for the cylinder but not for the slab; however, this resonance curve is open (has no maximum) if nonlinear damping is neglected (cf. the open resonance curve for a simple nonlinear oscillator if linear damping is neglected). Milner (1991) suggested that cubic damping closes the resonance curve for $0 < \epsilon - \delta = O(\delta^3)$ but overlooked the comparable role of cubic forcing (Miles 1993). It is implicit in these last two papers that the cubic damping is positive; in fact, cubic damping or cubic forcing or both may be negative (see below).

There remains the problem of pattern selection – i.e. the selection of a particular mode or combination of modes for the cylinder or the selection of both wavenumber and spatial pattern for the slab. The constraint $\beta = O(1)$ restricts the choice for the square cylinder to some linear combination of the degenerate, dominant modes $\cos kx$ and $\cos ky$; in particular, these basic modes describe orthogonal *rolls*‡, while $\cos kx + \cos ky$ and $\cos kx - \cos ky$ describe orthogonal *squares* with diagonal nodal lines. These patterns for the square cylinder also are available for the slab, but the wavenumber remains undetermined (in the present formulation) and the orientation is arbitrary (although it may be determined by distant boundaries). Moreover, hexagonal and other regular polygonal patterns may be possible for the slab, although they do not appear to have been observed. (The analysis of hexagonal symmetry requires three primary modes and six secondary modes, in contrast to two and three, respectively, for square symmetry.) Ezerskii *et al.* (1986) implicitly assume that, for capillary waves, squares dominate rolls. Milner (1991) allows for any regular polygonal pattern of capillary waves and concludes that squares dominate rolls and hexagons; however, he does not allow for detuning ($\beta \neq 0$), which, in the present analysis, implies parametric domains in which rolls are stable and squares are unstable. None of Ezerskii *et al.*, Milner, nor Miles (1993) provides for the instability of a particular mode with respect to perturbations of its complement – e.g. the instability of $\cos kx + \cos ky$ squares with respect to $\cos kx - \cos ky$ perturbations.

My formulation follows Miles (1976, 1984, 1993) and Miles & Henderson (1990,

† Simonelli & Gollub, Feng & Sethna and Umeki allow for detuning of this degeneracy by considering rectangular cylinders of aspect ratio close to one.

‡ The appellation *roll* for what an oceanographer would call a *straight-crested* wave is perhaps inappropriate for a surface wave, but it appears to be established and does offer the virtues of brevity and, in the present context, alphabetical proximity to *square*.

hereinafter referred to as MH). In §2, I pose a normal-mode expansion of the free-surface displacement that comprises the primary modes $\psi_1 = \sqrt{2}\cos kx$ and $\psi_2 = \sqrt{2}\cos ky$ and those secondary modes ψ_n for which $\langle \psi_l \psi_m \psi_n \rangle \neq 0$ ($\langle\,\rangle$ denotes an x, y-average) for $l, m = 1$ or 2. In §3, I choose these modes as the basis of an average-Lagrangian formulation, modified to incorporate linear damping, in which $\eta_n(t)$, the amplitudes of the ψ_n, are slowly modulated sinusoids with carrier frequency ω for the dominant modes and 2ω for the secondary modes. This leads to a set of canonical evolution equations for η_1 and η_2. I then obtain an alternative formulation in which the primary modes are $\psi_1 + \psi_2$ and $\psi_1 - \psi_2$ in place of ψ_1 and ψ_2, i.e. squares in place of rolls, through a canonical transformation of the Hamiltonian system. These alternative formulations are isomorphic, and the solution for rolls yields the solution for squares through a linear transformation of the coefficients in the Hamiltonian.

The fixed points (stationary solutions) of the evolution equations comprise: (i) the null solution; (ii) a pair of orthogonal roll patterns; (iii) a pair of orthogonal square patterns; (iv) coupled-mode solutions, for which both ψ_1 and ψ_2 are active and neither in phase nor in antiphase. I determine the resonance curves for, and the stability of, the rolls and squares in §4 and the coupled modes in §5. The null solution loses stability to either rolls or squares at a pair of pitchfork bifurcations. The fixed points for the rolls or squares lie on separate straight lines in an energy–frequency plane that terminate on these bifurcations. The two lines from the subcritical bifurcation are unstable (loci of unstable modes). Each of the two lines, R (rolls) and S (squares), from the supercritical bifurcation typically (the exceptions being associated with the proximity of a Wilton's-ripple resonance) comprises a stable and an unstable segment. The stable/unstable segment of R/S is contiguous to the null solution for $0 \leqslant \sigma < 0.210$ ($\sigma = 0$ for pure gravity waves) and conversely for $\frac{1}{3} < \sigma \leqslant 1$ ($\sigma = 1$ for pure capillary waves). The overlap between the stable segments of R and S is small, so that either rolls or squares tend to be definitely selected at a particular frequency. The bifurcations that separate the stable and unstable segments of R and S are connected by the locus of the coupled-mode solutions. The coupled-mode locus may have Hopf bifurcations, which suggests a possible transition to chaos (cf. Nagata 1991) without the spatial modulation postulated by Ezerskii *et al.* (1986) and Milner (1991) (although spatial modulation remains important in other respects).

The analysis in §§4 and 5 assumes $\epsilon - \delta \gg \delta^3$ and is not uniformly valid near the threshold $\epsilon = \delta$. In §6 I obtain uniformly valid approximations to the resonance curves on the assumption that cubic damping exceeds cubic forcing. The bifurcations from the null solution then are connected by two parabolas, one for rolls and one for squares. That bifurcation which is supercritical for $\epsilon - \delta \gg \delta^3$ remains supercritical for $\epsilon - \delta = O(\delta^3)$, but the second bifurcation may be either subcritical (in which case the parabola has a turning point) or supercritical. In the limiting case of deep-water capillary waves all roll solutions are unstable for $(\epsilon - \delta)/\delta^3 < 3.59$, while all square solutions except those on that segment of the parabola between the subcritical bifurcation and the turning point are stable for $(\epsilon - \delta)/\delta^3 < 8/9$. The resonant maxima tend to infinity, and each of the parabolas tends to the pair of straight lines determined in §4, for $\epsilon - \delta \gg \delta^3$. All of the coupled-mode solutions are unstable for $\epsilon - \delta = O(\delta^3)$.

The resonance curve does not close if cubic forcing exceeds cubic damping. Closure then may be possible through a fifth-order analysis, but the requisite analysis is intimidating.

The theoretical calculation of damping is unreliable except in rather special cases, such as a deep, laterally unbounded, uncontaminated fluid, which, following Milner (1991), I consider in Appendix A. The linear damping ratio may be defined as the measured threshold of ϵ, but I know of no measurements of cubical damping that are adequate for comparison with the present predictions.

I know of no experimental data that are adequate for a quantitative comparison with the present predictions of the stable domains of rolls and squares in either square cylinders or slabs. D. Henderson (personal communication) has observed squares, but not rolls, in the Faraday-resonant domain of the dominant mode in a square cylinder, but she did not undertake a systematic search of that domain. Crawford, Gollub & Lane (1993) report observations of stable squares and transient rolls in the resonant domains of the primary modes $\cos nkx$ and $\cos nky$ for $n = 2$ and 3, but their frequency range excluded $n = 1$.

Edwards & Fauve (1992) report stable rolls of roughly 1 cm length ($2\pi/k$) in a 12 cm circular slap of a glycerine solution with a kinematic viscosity of 0.7 $cm^2\,s^{-1}$ and an excitation frequency of $2\omega/2\pi = 60$ Hz; however, the corresponding Stokes damping ratio, $2\nu k^2/\omega$, is roughly 10^2, whereas the present formulation assumes $\delta \ll 1$.

Fauve *et al.* (1992) report stable rolls at the liquid–vapour interface of carbon dioxide close to its critical temperature, but the present analysis is not directly applicable to this configuration.

2. Normal modes

Following MH §2.1, we pose the free-surface displacement in the reference frame of the moving container in the form

$$\eta(\boldsymbol{x}, t) = \eta_n(t)\,\psi_n(\boldsymbol{x}; k_n), \tag{2.1}$$

where the ψ_n constitute a complete set of orthogonal modes, normalized according to $\langle \psi_m \psi_n \rangle = \delta_{mn}$, $\langle\ \rangle$ signifies an average over $\boldsymbol{x}$, δ_{mn} is the Kronocker delta, k_n are the modal wavenumbers, η_n are the modal amplitudes, and repeated dummy indices are summed over the participating modes except as noted. We choose

$$\psi_n = (2-\delta_{0j})^{\frac{1}{2}}(2-\delta_{0l})^{\frac{1}{2}}\cos jkx \cos lky, \quad k_n = (j^2+l^2)^{\frac{1}{2}}k, \tag{2.2a, b}$$

in which n comprehends the couplet (j, l), $j = 0, 1, \ldots$, $l = 0, 1, \ldots$; $\psi_0 = 1$ $(j = l = 0)$ is excluded by conservation of mass. Our primary modes are

$$\psi_1 = \surd 2 \cos kx, \quad \psi_2 = \surd 2 \cos ky \quad (k_1 = k_2 = k). \tag{2.3a, b}$$

The corresponding secondary modes, ψ_n, which are determined by the requirement

$$C_{lmn} \equiv \langle \psi_l \psi_m \psi_n \rangle \neq 0, \quad (l, m) = (1,1), (1,2), (2,1), (2,2), \tag{2.4}$$

are

$$\psi_3 = \surd 2 \cos 2kx, \quad \psi_4 = \surd 2 \cos 2ky, \quad \psi_5 = 2\cos kx \cos ky$$
$$(k_3 = k_4 = 2k, k_5 = \surd 2 k), \tag{2.5a–c}$$

for which

$$C_{11n} = \frac{1}{\surd 2}\delta_{n3}, \quad C_{22n} = \frac{1}{\surd 2}\delta_{n4}, \quad C_{12n} = C_{21n} = \delta_{n5}. \tag{2.6a–c}$$

The primary modes ψ_1 and ψ_2 describe rolls with crests parallel to the y- and x-axes, respectively. The composite modes (cf. Rayleigh 1883)

$$\psi_\pm \equiv \frac{1}{\surd 2}(\psi_1 \pm \psi_2) = \cos kx \pm \cos ky = \begin{cases} 2\cos \hat{k}\hat{x} \cos \hat{k}\hat{y} \\ 2\sin \hat{k}\hat{x} \sin \hat{k}\hat{y} \end{cases}, \tag{2.7a, b}$$

where $$\hat{k} = \frac{k}{\surd 2}, \quad \hat{x} = \frac{x+y}{\surd 2}, \quad \hat{y} = \frac{y-x}{\surd 2}, \tag{2.8 a–c}$$

describe orthogonal square patterns with axes inclined at $\frac{1}{4}\pi$ to those of x and y and have diagonal nodes in the square $0 < x,\ y < \pi/k$.

3. Evolution equations ($kd \gg 1$)

We pose the slowly modulated amplitudes of the primary modes in the form

$$\eta_n = 2\epsilon^{\frac{1}{2}}k^{-1}\tanh kd[p_n(\tau)\cos\omega t + q_n(\tau)\sin\omega t] \quad (n = 1,2), \quad \tau = \epsilon\omega t, \tag{3.1 a, b}$$

where $\epsilon \equiv ka_0 \tanh kd$, (1.2*a*). Proceeding as in Miles (1976, 1984) and MH §2, we construct (in Appendix A) the average Lagrangian (A 14) and incorporate linear damping to obtain the evolution equations

$$\dot{p}_n = -\frac{\partial D}{\partial p_n} - \frac{\partial H}{\partial q_n}, \quad \dot{q}_n = -\frac{\partial D}{\partial q_n} + \frac{\partial H}{\partial p_n}, \tag{3.2 a, b}$$

where, here and subsequently, error factors of $1 + O(\epsilon)$ are implicit, $\dot{p}_n \equiv \mathrm{d}p_n/\mathrm{d}\tau$,

$$D = \alpha_n e_n, \quad H = \tfrac{1}{2}(p_n p_n - q_n q_n) + \beta_n e_n + A_{mn} e_m e_n + \tfrac{1}{2}Br^2, \tag{3.3 a, b}$$

$$e_n \equiv \tfrac{1}{2}(p_n^2 + q_n^2), \quad r \equiv p_1 q_2 - p_2 q_1, \tag{3.4 a, b}$$

$\alpha_1 = \alpha_2 = \alpha$ and $\beta_1 = \beta_2 = \beta$ are defined by (1.2*b, c*), $A_{11} = A_{22} \equiv A$, $A_{12} = A_{21} \equiv C$, and B are given by (A 17), repeated indices now are summed only over 1 and 2, and we have omitted cubic damping and forcing (which are incorporated in §6). The term $\frac{1}{2}(p_n p_n - q_n q_n)$ in (3.3*b*) is derived from the potential energy associated with the acceleration $\ddot{z}_0$ and represents the parametric excitation, the term $\beta_n e_n$ represents the residue of the second-order (in amplitude) components of the Lagrangian after allowing for the proximity to resonance, and the remaining terms represent the quartic components ($\frac{1}{2}Br^2$ is the dimensionless angular momentum of the fluid motion, which enters the calculation through the kinetic energy).

The deep-water limits ($kd \uparrow \infty$) of (A 17) yield (see figure 1)

$$A = \tfrac{1}{2} + \tfrac{9}{8}\sigma + \frac{1}{1+3\sigma} - \frac{\frac{1}{2}}{1-3\sigma}, \quad B = -2 - \tfrac{1}{4}\sigma - \frac{1}{1+\sigma}, \tag{3.5 a, b}$$

$$C = \surd 2 - 1 + \tfrac{3}{4}\sigma + \frac{1}{1+\sigma} - \frac{\frac{1}{2}(3-2\surd 2)^2}{2\surd 2 - 1 - \sigma}, \tag{3.5 c}$$

where $$\sigma = k^2 l_*^2/(1 + k^2 l_*^2) \tag{3.6}$$

is a measure of capillarity. B varies monotonically from -3 to -2.75 and C from 1.40 to 1.65 as σ increases from 0 (gravity waves) to $\sigma = 1$ (capillary waves). The corresponding limits of A are 1 and 2.13, but $A = \infty$ at $\sigma = \frac{1}{3}$, near which the present formulation (in particular, the hypothesis that the primary modes 1 and 2 dominate the secondary modes 2 and 4) fails owing to the resonances between modes 1 and 3 ($k_3 = 2k_1$ and $\omega_3 = 2\omega_1$, corresponding to Wilton's ripples) and 2 and 4, which we exclude.

The canonical transformation

$$\surd 2 p_\pm = p_1 \pm p_2, \quad \surd 2 q_\pm = q_1 \pm q_2, \tag{3.7 a, b}$$

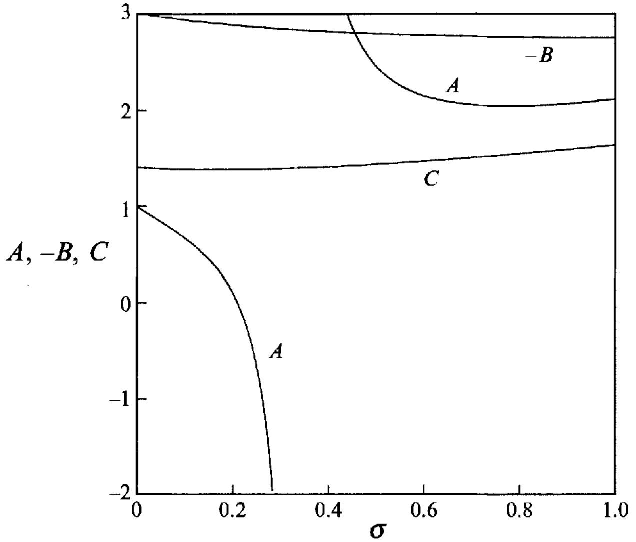

FIGURE 1. A, B and C for $kd \gg 1$, as given by (3.5).

where alternative signs are vertically ordered, implies, through (2.1), (2.7) and (3.1),

$$\surd 2\eta_{\pm}(t) = \eta_1(t) \pm \eta_2(t), \quad \eta_1 \psi_1 + \eta_2 \psi_2 = \eta_+ \psi_+ + \eta_- \psi_- \tag{3.8a, b}$$

for the primary modes in (2.1) and carries (3.2)–(3.4) over to isomorphic forms in which p_1, p_2, q_1, q_2, A, B, C, respectively, are replaced by p_+, p_-, q_+, q_-,

$$A_+ = \tfrac{1}{2}(A+C), \quad B_+ = B+C-A, \quad C_+ = \tfrac{1}{2}(3A-C). \tag{3.9a–c}$$

The introduction of action-angle variables through the canonical transformation

$$p_n = (2e_n)^{\frac{1}{2}} \cos\theta_n, \quad q_n = (2e_n)^{\frac{1}{2}} \sin\theta_n \tag{3.10a, b}$$

yields

$$\dot{e}_n = -2e_n \frac{\partial D}{\partial e_n} - \frac{\partial H}{\partial e_n}, \quad \dot{\theta}_n = \frac{\partial H}{\partial e_n} \tag{3.11a, b}$$

and

$$H = e_n \cos 2\theta_n + \beta(e_1+e_2) + Ae_n e_n + 2e_1 e_2[B\sin^2(\theta_1-\theta_2)+C]. \tag{3.12}$$

Substituting (3.3*a*) and (3.12) into (3.11), we obtain

$$\dot{e}_1 + 2\alpha e_1 = 2e_1 \sin 2\theta_1 - 2Be_1 e_2 \sin 2(\theta_1-\theta_2), \tag{3.13a}$$

$$\dot{e}_2 + 2\alpha e_2 = 2e_2 \sin 2\theta_2 + 2Be_1 e_2 \sin 2(\theta_1-\theta_2), \tag{3.13b}$$

$$\dot{\theta}_1 = \beta + \cos 2\theta_1 + 2Ae_1 + 2[B\sin^2(\theta_1-\theta_2)+C]\, e_2 \tag{3.13c}$$

and

$$\dot{\theta}_2 = \beta + \cos 2\theta_2 + 2Ae_2 + 2[B\sin^2(\theta_1-\theta_2)+C]\, e_1. \tag{3.13d}$$

4. Single-mode fixed points

The fixed points of (3.13) comprise: (i) the null solution $e_1 = e_2 = 0$, which is stable if and only if $\beta^2 > 1-\alpha^2$; (ii) rolls for which $e_2 = 0$ and their complements for which $e_1 = 0$; (iii) squares for which $e_2 = e_1$ and $\theta_1 - \theta_2 = 0$ and their complements for which

Regime	σ	A	$A+C$	$A-B-C$	K	e_c	K_+	e_{c+}
(i)	(0, 0.210)	+	+	+	+	+	−	+
(ii)	(0.210, 0.2729)	−	+	+	−	−	−	+
(iii)	(0.2729, 0.2742)	−	−	+	−	−	+	−
(iv)	(0.2742, $\frac{1}{3}$)	−	−	−	−	+	+	−
(v)	($\frac{1}{3}$, 1)	+	+	+	−	+	+	+

TABLE 1. Parametric domains for single-mode fixed points for $kd \gg 1$. ($\sigma = 0.210, 0.2729$ and 0.2742 are determined by the zeros of A, $A+C$ and $A-B-C$, respectively, and $\sigma = \frac{1}{3}$ is determined by the resonance between the primary mode and its second harmonic.)

$\theta_1-\theta_2 = \pm\pi$; (iv) coupled-mode solutions, for which $e_{1,2} > 0$ and $\theta_1-\theta_2 \neq 0$ or $\pm\pi$. This section deals with the single-mode fixed points (ii) and (iii); coupled-mode fixed points are considered in §5.

Letting $\dot{e}_1 = \dot{\theta}_1 = e_2 = 0$ in (3.13*a, c*), we obtain

$$e_1 = |2A|^{-1}(\pm\beta_*-\beta\mathscr{S}), \quad \sin 2\theta_1 = \alpha, \quad \cos 2\theta_1 = \mp\beta_*\mathscr{S}, \qquad (4.1\,a\text{–}c)$$

where

$$\beta_* \equiv (1-\alpha^2)^{\frac{1}{2}}, \quad \mathscr{S} \equiv \operatorname{sgn} A, \qquad (4.2\,a, b)$$

and, here and subsequently, the upper/lower alternatives correspond to the upper/lower branches of the resonance curve in a (β, e)-plane ($e \equiv e_1+e_2$). The upper/lower branch joins the null solution at a super/subcritical pitchfork bifurcation at $\beta = \pm\beta_*\mathscr{S}$. We refer to this solution as R_1 and to its complement $(1 \leftrightarrows 2)$ as R_2.

A linear stability analysis (cf. Nagata 1989), based on small perturbations of p_n and q_n proportional to $\exp(\lambda\tau)$, implies that mode-1 and mode-2 perturbations are linearly independent and yields the respective characteristic equations

$$\lambda^2+2\alpha\lambda+\Lambda_n = 0 \quad (n = 1 \text{ or } 2), \qquad (4.3)$$

where

$$\Lambda_1 = \pm 8|A|\beta_* e, \quad \Lambda_2 = 4K\beta_* e[\pm 1-(e/e_c)], \qquad (4.4\,a, b)$$

$$K \equiv (C-A)\mathscr{S}, \quad e_c \equiv \beta_*(A-B-C)^{-1}\mathscr{S}. \qquad (4.5\,a, b)$$

Stability requires $\Lambda_n > 0$. It follows from (4.4*a*) that the upper/lower branch of the resonance curve (4.1*a*) is stable/unstable with respect to mode-1 perturbations. It follows from (4.4*b*) that if $K \gtrless 0$ the upper branch is stable/unstable with respect to mode-2 perturbations for $e/e_c < 1$ or unstable/stable for $e/e_c > 1$ (e_c need not be positive, but $e < 0$ is inadmissible).

The fixed points of (3.13) for $e_2 = e_1$ and $\theta_2 = \theta_1$ and their stability are determined by (4.1)–(4.5) through the canonical transformation (3.7) and $(A, B, C) \rightarrow (A_+, B_+, C_+)$, where $A_+, \ldots$ are given by (3.9). In particular,

$$e_+ = |A+C|^{-1}(\pm\beta_*-\beta\mathscr{S}_+), \quad \mathscr{S}_+ \equiv \operatorname{sgn}(A+C), \qquad (4.6\,a, b)$$

$$K_+ = (A-C)\mathscr{S}_+, \quad e_{c+} = -B^{-1}\beta_*\mathscr{S}_+. \qquad (4.7\,a, b)$$

We refer to this solution as S_+ and to its complement as S_-.

The parametric domains of the above modes for $kd \gg 1$, so that A, B and C are given by (3.5), are listed in table 1. Resonance curves for $\sigma = 0$ and 1, which are representative of regimes (i) and (v) of table 1, are plotted in figure 2.

The results in this section reduce to those of Nagata (1989) for his 'single' (roll) and 'mixed' (square) modes if $\sigma = 0$, for which he gives comprehensive stability graphs for finite *kd*.

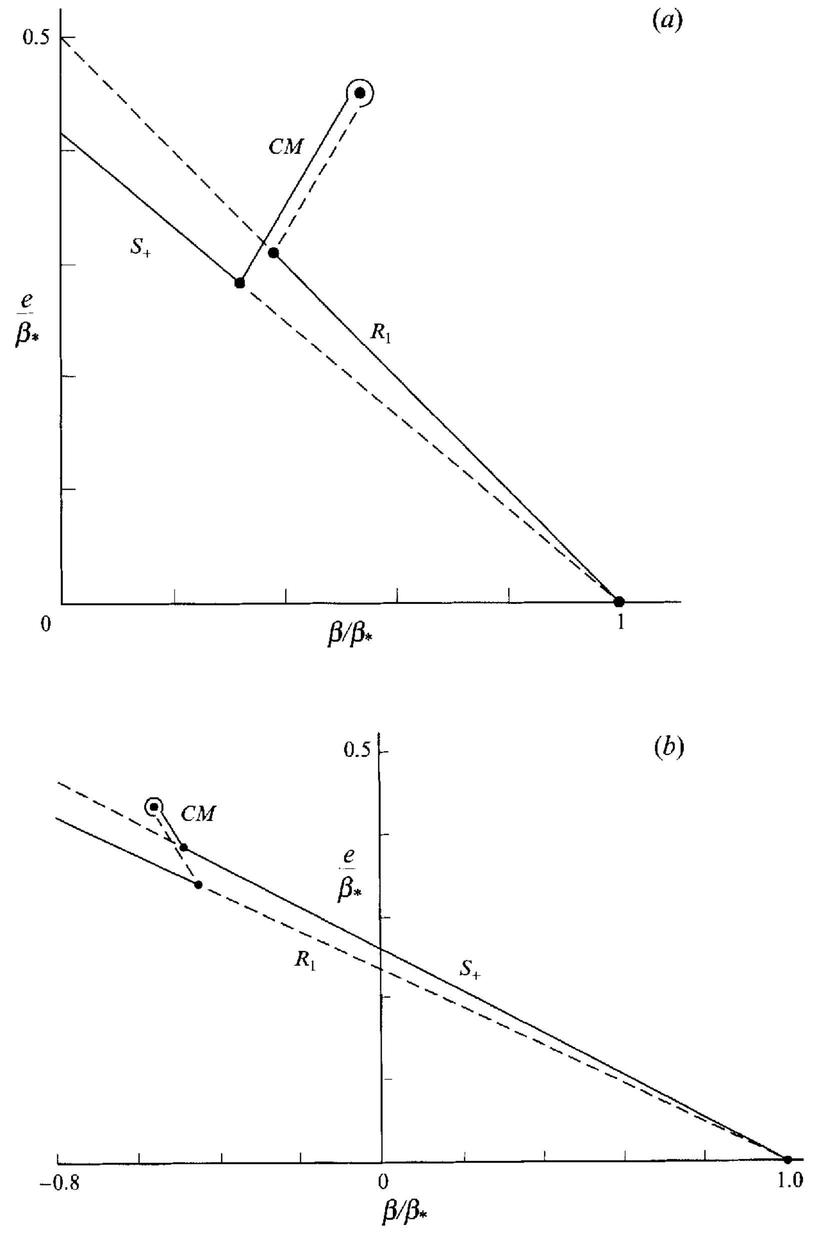

FIGURE 2. The upper branches of the resonance curves R_1, S_+ and CM, as determined from (4.1 *a*), (4.6 *a*) and (5.1 *c*), respectively, for $\alpha = 0.1$ and (*a*) $\sigma = 0$, (*b*) $\sigma = 1$. The solid/dashed segments are stable/unstable. The extensions of CM above R_1 in (*a*) and S_+ in (*b*) disappear for $\alpha > 0.260$ and $\alpha > 0.272$, respectively.

5. Coupled-mode fixed points

The fixed points of (3.13) for $e_{1,2} > 0$ are determined by (cf. Nagata 1989)

$$e \equiv e_1 + e_2 = \frac{[\mu^2\gamma^2 - \alpha^2(1+\mu-\gamma^2)^2]^{\frac{1}{2}}}{|A-C|\gamma^2}, \quad f \equiv e_1 - e_2 = \frac{\alpha(1-\gamma^2)^{\frac{1}{2}}}{(A-C)\gamma}, \qquad (5.1\,a, b)$$

$$\beta = -(A+B+C)\,e, \quad \cos 2\theta_{1,2} = B[\gamma^2 e \pm (1+\mu-\gamma^2)f], \qquad (5.1\,c, d)$$

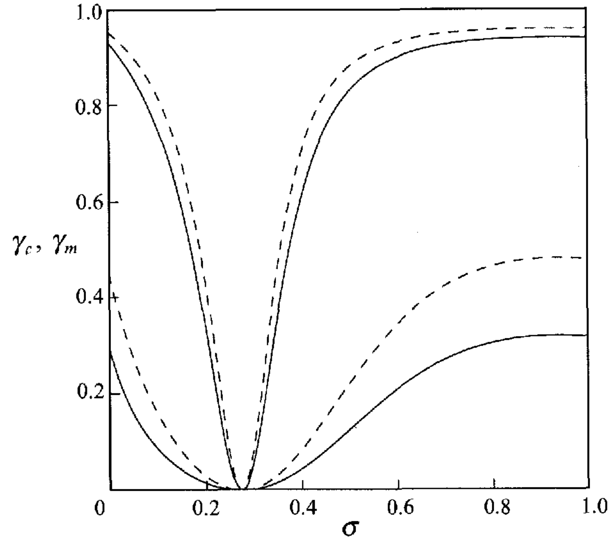

FIGURE 3. The terminal values γ_c^2 (5.3) (——) and γ_m^2 (5.4*b*) (---) for $\alpha = 0.1$ (lower curves) and $\alpha = \frac{1}{2}$ (upper curves).

where
$$\gamma \equiv \cos(\theta_1 - \theta_2), \quad \mu \equiv (C-A)/B. \tag{5.2a, b}$$
It suffices to choose $\operatorname{sgn}\gamma = \operatorname{sgn}(A-C)$, so that $e_2 < e_1$ (the complementary solution, for which $e_2 > e_1$, may be obtained by interchanging the subscripts 1 and 2), and to regard γ^2 as a control parameter with the admissible range $[\gamma_c^2, 1]$, where (see figure 3)
$$\gamma_c^2 = \frac{(1+\mu)^2\alpha^2}{(1+\mu)^2\alpha^2 + \mu^2(1-\alpha^2)} \tag{5.3}$$
is determined by the constraint $e_2 \geqslant 0$ $(e > f)$.

The locus CM of the coupled-mode fixed points in the (β, e_1, e_2)-space projects on the straight line (5.1*c*) in the (β, e)-plane and terminates at pitchfork bifurcations at $e = e_c$ $(\gamma = \gamma_c)$ on R_1 and $e = e_{c+}$ $(\gamma = 1)$ on S_+ (see figure 2). It has a turning-point maximum at
$$e = e_m = \frac{(\mu^2 + 4\alpha^2\mu + 4\alpha^2)^{\frac{1}{2}}}{2|B|(1+\mu)\alpha} \quad \text{for} \quad \gamma^2 = \gamma_m^2 \equiv \frac{2(1+\mu)^2\alpha^2}{2(1+\mu)\alpha^2 + \mu^2} \tag{5.4a, b}$$
if and only if $\gamma_c^2 < \gamma_m^2 < 1$ (see figure 3), which requires either (see figure 4)
$$\alpha^2 < \frac{\frac{1}{2}\mu}{1+\mu} \equiv \alpha_+^2 \ (\mu < -2 \text{ or } \mu > 0) \quad \text{or} \quad \alpha^2 < -\tfrac{1}{2}\mu \equiv \alpha_1^2 \quad (-2 < \mu < 0). \tag{5.5a, b}$$
CM then has two branches; CM_1, which is traversed from the pitchfork bifurcation at $e = e_c$ on R_1 to the turning point at $e = e_m$ as γ increases from γ_c to γ_m, and CM_+, which is traversed from the turning point to the pitchfork bifurcation at $e = e_{c+}$ on S_+ as γ increases from γ_m to 1. The turning point coalesces with the pitchfork bifurcation on S_+ for $\gamma_m = 1$ $(\alpha = \alpha_+)$, and if $\gamma_m > 1$ the single branch CM_1 is traversed from e_c to e_{c+} as γ^2 increases from γ_c^2 to 1. The turning point coalesces with the pitchfork bifurcation on R_1 if $\gamma_m = \gamma_c$ $(\alpha = \alpha_1)$, and if $\gamma_m < \gamma_c$ the single branch CM_+ is traversed from e_c to e_{c+} as γ^2 increases from γ_c^2 to 1.

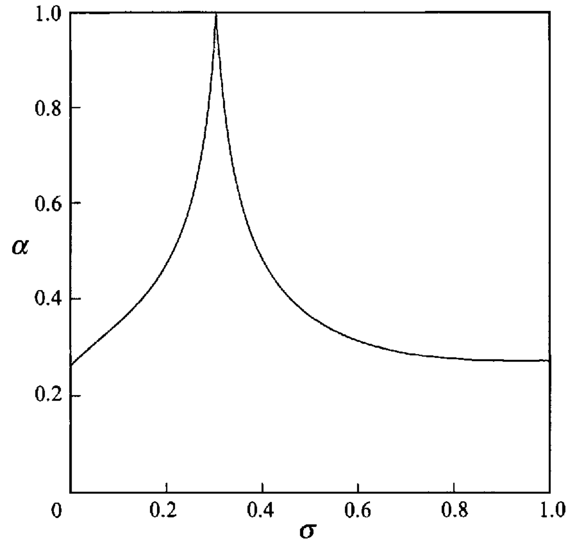

FIGURE 4. The critical values of α, as given by (5.5*a*/*b*) to the right/left of the cusp.

The stability determinant for small perturbations about the fixed point(s) (5.1) is

$$\Delta(\lambda) = \lambda^4 + 4\alpha\lambda^3 + (2b+4\alpha^2)\lambda^2 + 4\alpha b\lambda + c \tag{5.6a}$$

$$= [\lambda^2 + 2\alpha\lambda + b + (b^2-c)^{\frac{1}{2}}][\lambda^2 + 2\alpha\lambda + b - (b^2-c)^{\frac{1}{2}}], \tag{5.6b}$$

where

$$b = \frac{-2[A-B-C+\gamma^2(B+2C)](1-\alpha^2)(\gamma^2-\gamma_c^2)}{B(1-\gamma_c^2)\gamma^4} + \frac{4A(A-B-C)\alpha^2(1-\gamma^2)}{(A-C)^2\gamma^2} \tag{5.7a}$$

and

$$c = 32\left(\frac{A+B+C}{A-C}\right)\left(\frac{A-B-C}{B}\right)^2 \frac{\alpha^2(1-\alpha^2)(\gamma^2-\gamma_c^2)(\gamma^2-\gamma_m^2)(1-\gamma^2)}{(1-\gamma_c^2)\gamma_m^2\gamma^6}. \tag{5.7b}$$

The necessary and sufficient conditions for stability are

$$b \geqslant 0, \quad c \geqslant 0, \quad h \equiv b^2 + 4\alpha^2 b - c \geqslant 0. \tag{5.8a–c}$$

The condition $h = 0$ with $b, c > 0$ implies a Hopf bifurcation.

If (as we henceforth assume) $kd \gg 1$ $(A+B+C)/(A-C) > 0$ for all σ, and c changes sign at $\gamma = \gamma_c (e = e_c)$, $\gamma = \gamma_m (e = e_m)$ and $\gamma = 1 (e = e_{c+})$. If either

$$\gamma_c^2 < \gamma^2 < \gamma_m^2 \text{ or } \gamma_m^2 > 1 \; c < 0,$$

and all of CM_1 is unstable. We therefore need consider further only the stability of CM_+, for which either $\gamma_m^2 < \gamma_c^2 < \gamma^2 < 1$ or $\gamma_c^2 < \gamma_m^2 < \gamma^2 < 1$, so that $c > 0$.

It follows from (5.7*a*) that (for $\gamma_c^2 < \gamma^2 < 1$) $b > 0$ if $A > 0$ ($\sigma < 0.210$ or $\sigma > \frac{1}{3}$) and $b < 0$ if $A - B - C < 0$ ($0.2742 < \gamma^2 < \frac{1}{3}$); accordingly, $b = 0$ is possible only if $0.210 < \sigma < 0.2742$, in which domain $|(A-B-C)-(A+C)| = |B+2C| \lesssim 10^{-2}$ and we may approximate (5.7*a*) by

$$b = \left(\frac{A+C}{C}\right)\left[\frac{(1-\alpha^2)(\gamma^2-\gamma_c^2)}{(1-\gamma_c^2)\gamma^4} + \frac{4AC\alpha^2(1-\gamma^2)}{(A-C)^2\gamma^2}\right] \quad (B+2C \approx 0). \tag{5.9}$$

This admits the single zero

$$\gamma^2 = \frac{2\gamma_c^2}{1+\kappa+[(1+\kappa)^2-4\kappa\gamma_c^2]^{\frac{1}{2}}} \equiv \gamma_b^2, \quad \kappa \equiv \frac{4AC\alpha^2(1-\gamma_c^2)}{(A-C)^2(1-\alpha^2)} \tag{5.10a, b}$$

(so that $b < 0$ for $\gamma_c < \gamma < \gamma_b$) if and only if $A < 0$.

Analytical approximations to the zeros of h appear to be possible only if $\alpha^2 \ll \gamma^2 < 1$, in which domain

$$\gamma^4 h = 4\left(\frac{A-B-C}{B}\right)^2\left\{\left[1+\left(\frac{B+2C}{A-B-C}\right)\gamma^2\right]^2 - \frac{4(A-C)(A+B+C)}{(A-B-C)^2}\gamma^2(1-\gamma^2)\right\}+O(\alpha^2). \tag{5.11}$$

A necessary condition for the zeros of (5.11) to be real is $A < 0$. Invoking $|B+2C| \ll 1$ in this domain (see above), we obtain the approximations

$$C^2\gamma^4 h = (A+C)^2 - 4(A-C)^2\gamma^2(1-\gamma^2) + O(\alpha^2) \tag{5.12}$$

and

$$\gamma_{h_\pm}^2 = \tfrac{1}{2} \pm (C-A)^{-1}(-AC)^{\frac{1}{2}} + O(\alpha^2) \tag{5.13}$$

for the Hopf bifurcations. Note that each of γ_b^2, γ_c^2 and γ_m^2 is $O(\alpha^2)$, and hence that $b, c > 0$, for $\gamma_{h-}^2 < \gamma^2 < \gamma_{h+}^2$. The zeros γ_{h+} and γ_{h-} coalesce, and the Hopf bifurcations disappear, as $A \uparrow 0$; they tend to 1 and 0, respectively, as $A+C \to 0$ (but the assumption $\gamma^2 \ll \alpha^2$ then is violated by γ_{h-}^2). A numerical investigation suggests, but I have not proved, that $h > 0$ for all α in $(0,1)$ if $A > 0$ – i.e. for either $0 < \sigma < 0.210$ or $\frac{1}{3} < \sigma < 1$. But Nagata's (1989, 1991) numerical examples imply that Hopf bifurcations exist for $\sigma = 0$ and $kd \approx 1$.

6. Threshold analysis

Cubic forcing and damping are significant in the threshold domain $0 < 1-\alpha = O(\epsilon^2)$ or, equivalently, $\beta_* = O(\epsilon)$, in which first-order forcing and damping almost balance. The approximations (3.3) and (3.12) then must be augmented to obtain (cf. Miles 1993)

$$D = \alpha_n e_n + \tilde{A}_{mn} e_m e_n = \alpha(e_1+e_2) + \tilde{A}(e_1^2+e_2^2) + 2\tilde{C}e_1 e_2 \tag{6.1}$$

and

$$H = e_n \cos 2\theta_n + \beta(e_1+e_2) + Ae_n e_n + 2e_1 e_2[B\sin^2(\theta_1-\theta_2)+C] + 2P(e_1^2\cos 2\theta_1 + e_2^2\cos 2\theta_2) + 2Qe_1 e_2(\cos 2\theta_1 + \cos 2\theta_2), \tag{6.2}$$

where: $\tilde{A}$ and $\tilde{C}$ are derived in Appendix B under rather limiting assumptions but may have to be determined empirically; $P_{11} = P_{22} \equiv P$ and $P_{12} = P_{21} \equiv Q$ are given by (A 18) and have the limiting forms

$$P_{11} = P_{22} \equiv P = \frac{2\epsilon}{1-9\sigma^2}, \quad P_{12} = P_{21} \equiv Q = \frac{2(2\sqrt{2}-3)\epsilon}{(1+\sigma)(2\sqrt{2}-1-\sigma)} \quad (kd \gg 1). \tag{6.3a, b}$$

Substituting (6.1) and (6.2*b*) into (3.11), we obtain (cf. (3.13))

$$\dot{e}_1 + 2\alpha e_1 = 2e_1[(1+2Pe_1+2Qe_2)\sin 2\theta_1 - Be_2\sin 2(\theta_1-\theta_2) - 2\tilde{A}e_1 - 2\tilde{C}e_2], \tag{6.4a}$$

$$\dot{\theta}_1 = \beta + (1+4Pe_1)\cos 2\theta_1 + 2Qe_2(\cos 2\theta_1 + \cos 2\theta_2) + 2Ae_1 + 2[B\sin^2(\theta_1-\theta_2)+C]e_2, \tag{6.4b}$$

and their complements, which follow from the interchange of the subscripts 1 and 2.

We now re-scale (6.4) according to

$$e_n = \beta_* \mathscr{E}_n(T), \quad \theta_n = \tfrac{1}{4}\pi + \beta_* \Theta_n(T), \quad T = \beta_*^2 \tau, \quad \mathscr{B} = \beta/\beta_*, \tag{6.5a–d}$$

and let $1-\alpha \downarrow 0$ to obtain (where $\dot{\mathscr{E}}_n \equiv \mathrm{d}\mathscr{E}_n/\mathrm{d}T$)

$$\dot{\mathscr{E}}_1 = \mathscr{E}_1\{1-4[\hat{A}\mathscr{E}_1+\hat{C}\mathscr{E}_2+B\mathscr{E}_2(\Theta_1-\Theta_2)+\Theta_1^2]\}, \quad \Theta_1 = \tfrac{1}{2}\mathscr{B}+A\mathscr{E}_1+C\mathscr{E}_2, \qquad (6.6a, b)$$

and their complements, where error factors of $1+O(1-\alpha)$ are implicit and

$$\hat{A} \equiv (\tilde{A}-P)/\beta_*, \quad \hat{C} \equiv (\tilde{C}-Q)/\beta_*. \qquad (6.7a, b)$$

Eliminating Θ_1 and Θ_2 from (6.6a, b) and their complements, we obtain

$$\dot{\mathscr{E}}_1 = \mathscr{E}_1[1-(\mathscr{B}+2A\mathscr{E}_1+2C\mathscr{E}_2)^2-4(A-C)\,B(\mathscr{E}_1-\mathscr{E}_2)\,\mathscr{E}_2-4\hat{A}\mathscr{E}_1-4\hat{C}\mathscr{E}_2] \qquad (6.8a)$$

and

$$\dot{\mathscr{E}}_2 = \mathscr{E}_2[1-(\mathscr{B}+2A\mathscr{E}_2+2C\mathscr{E}_1)^2+4(A-C)\,B(\mathscr{E}_1-\mathscr{E}_2)\,\mathscr{E}_1-4\hat{A}\mathscr{E}_2-4\hat{C}\mathscr{E}_1]. \qquad (6.8b)$$

The fixed points of (6.8) may be classified as in §4. Rolls are determined by (cf. (4.1))

$$(\mathscr{B}+2A\mathscr{E})^2+4\hat{A}\mathscr{E} = 1 \quad (\mathscr{E}_2 = 0), \qquad (6.9)$$

and its complement, to which we refer as R_1 and R_2. It suffices to consider $A > 0$, since changing the sign of A is equivalent to changing the sign of $\mathscr{B}$. If $\hat{A} > 0$ (third-order damping exceeds third-order forcing) (6.9) is a parabola in a $(\mathscr{B}, \mathscr{E})$-plane (see figure 5), joins the null solution at a supercritical pitchfork bifurcation at $\mathscr{B} = 1$, has a maximum at

$$\mathscr{B} = -\tfrac{1}{2}A\hat{A}^{-1}, \quad \mathscr{E} = \tfrac{1}{4}\hat{A}^{-1} \equiv \mathscr{E}_{\max}, \qquad (6.10a, b)$$

and re-joins the null solution at a pitchfork bifurcation at $\mathscr{B} = -1$ that is sub/supercritical for $\hat{A} \lessgtr A$. The subcritical bifurcation is accompanied by a turning point at

$$\mathscr{B} = -\tfrac{1}{2}(A\hat{A}^{-1}+A^{-1}\hat{A}), \quad \mathscr{E} = \tfrac{1}{4}(\hat{A}^{-1}-A^{-2}\hat{A}) \quad (\hat{A} < A). \qquad (6.11a, b)$$

If $|\hat{A}| \ll 1$ ($\beta_* \gg \epsilon$), $\hat{A}\mathscr{E}$ may be neglected in (6.9), which then comprises the two straight lines of (4.1a).

If $\hat{A} < 0$ (third-order forcing exceeds third-order damping) the parabola (6.9) does not close in $\mathscr{E} > 0$. It is possible that the retention of fifth-order forcing and damping could close the resonance curve for $\hat{A} < 0$, but the analysis would be both tedious and, for the damping, of questionable significance.

Small mode-1 and mode-2 perturbations of (6.9) are linearly independent, as in §4. The solution is stable with respect to mode-1 perturbations except for states between the subcritical bifurcation at $\mathscr{B} = -1$ and the turning point (6.11), which are unstable. The stability with respect to mode-2 perturbations proportional to $\exp(\lambda T)$ is determined by

$$\lambda = 1-\mathscr{B}^2-4(\hat{C}+C\mathscr{B})\,\mathscr{E}+4(AB-BC-C^2)\,\mathscr{E}^2 \qquad (6.12a)$$

$$= 4[\hat{A}-\hat{C}\pm(A-C)\,(1-4\hat{A}\mathscr{E})^{\frac{1}{2}}]\,\mathscr{E}-4(A-C)\,(A-B-C)\,\mathscr{E}^2, \qquad (6.12b)$$

where the upper/lower sign in (6.12b) corresponds to points on the right/left of the maximum (6.10). The transition points, at which $\lambda = 0$, correspond to the intersections of R_1 and CM (see (6.22a) below) and are given by (cf. (4.5b))

$$\mathscr{E} = (A-B-C)^{-1}\{\mathscr{C}\mathscr{D}\pm[1-\mathscr{C}^2(1-\mathscr{D}^2)]^{\frac{1}{2}}\} \equiv \mathscr{E}_{c\pm}, \qquad (6.13)$$

where

$$\mathscr{C} = \frac{\hat{A}-\hat{C}}{A-C}, \quad \mathscr{D} = 1-\left(\frac{A-C}{A-B-C}\right)\left(\frac{2\hat{A}}{\hat{A}-\hat{C}}\right), \qquad (6.14a, b)$$

and $\mathscr{E}_{c-} > 0$ if and only if $\mathscr{C} > 1$. $\mathscr{C}$ is proportional to, while $\mathscr{D}$ is independent of ϵ/β_*. We assume that $\mathscr{C} > 0$, for which $kd \gg 1$ and $\sigma > \frac{1}{3}$ are sufficient. The upper transition point $\mathscr{E}_{c+}$ moves from e_c/β_* (see (4.5)) for $\mathscr{C} = 0$ through the maximum (6.10) for

$$\mathscr{C} = [2(1-\mathscr{D})]^{-\frac{1}{2}} \equiv \mathscr{C}_m \qquad (6.15)$$

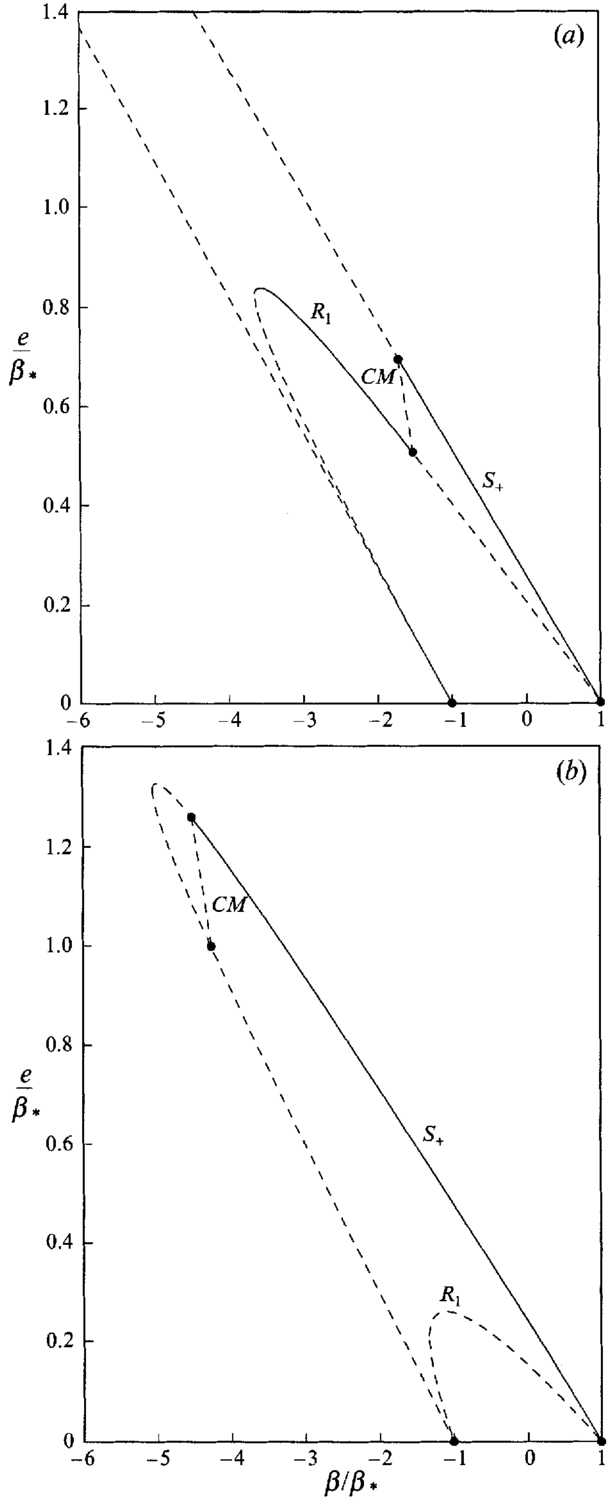

FIGURE 5. The threshold resonance curve for rolls (R_1), squares (S_+) and coupled modes (CM) with $\sigma = 1$ and (*a*) $\epsilon/\beta_* = 0.266$ ($\mathscr{C} = 1$) and (*b*) $\epsilon/\beta_* = 0.8$. The solid/dashed segments comprise stable/unstable states.

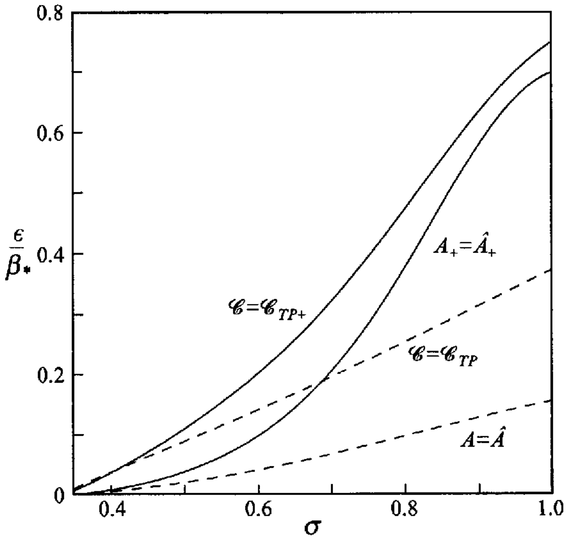

FIGURE 6. The critical values of ϵ/β_* for which $\mathscr{C} = \mathscr{C}_{TP}$, $A = \hat{A}$, $\mathscr{C} = \mathscr{C}_{TP+}$, and $A_+ = \hat{A}_+$.

and (if $\hat{A} < A$) the turning point (6.11) for

$$\mathscr{C} = (1-\mathscr{D})^{-1}\left[\frac{1+\mathscr{D}}{1-\mathscr{D}}+\left(\frac{A+B+C}{2A}\right)^2\right]^{-\frac{1}{2}} \equiv \mathscr{C}_{TP} \tag{6.16}$$

and joins the lower transition point $\mathscr{E}_{c-}$ for

$$\mathscr{C} = (1-\mathscr{D}^2)^{-\frac{1}{2}} \equiv \mathscr{C}_* \quad (\mathscr{D} < 1), \tag{6.17}$$

at which point CM is tangent to R_1. All of R_1 is unstable for $\mathscr{C} > \mathscr{C}_{TP}$, and CM does not intersect R_1 if $\mathscr{C} > \mathscr{C}_*$. If $\hat{A} > A$ the turning point disappears, and all of R_1 is unstable if $\mathscr{C} > 1$. The values of ϵ/β_* at which $\mathscr{C} = \mathscr{C}_{TP}$ (so that all of R_1 is unstable) and $\hat{A} = A$ are plotted in figure 6.

The preceding results for rolls may be transformed to the corresponding results for squares through (3.7), (3.9) and (note that $P_+ = P$ in Miles (1993) with $kd = \infty$ therein)

$$P_+ = \tfrac{1}{2}(P+Q), \quad Q_+ = \tfrac{1}{2}(3P-Q), \tag{6.18\textit{a, b}}$$

as in §4. In particular, the counterpart of (6.13) is (cf. (4.7*b*))

$$\mathscr{E} = -B^{-1}\{\mathscr{C}\mathscr{D}_+ \pm [1-\mathscr{C}^2(1-\mathscr{D}_+^2)]^{\frac{1}{2}}\}, \tag{6.19}$$

where $\mathscr{C}$ is given by (6.14*a*), and

$$\mathscr{D}_+ \equiv 1+\left(\frac{A-C}{B}\right)\left(\frac{\hat{A}+\hat{C}}{\hat{A}-\hat{C}}\right). \tag{6.20}$$

The entire upper branch, from the pitchfork bifurcation at $\mathscr{B} = 1$ and $\mathscr{E} = 0$ to the turning point, is stable if

$$\mathscr{C} > (1-\mathscr{D}_+)^{-1}\left[\frac{1+\mathscr{D}_+}{1-\mathscr{D}_+}+\left(\frac{A+B+C}{A+C}\right)^2\right]^{-\frac{1}{2}} \equiv \mathscr{C}_{TP+}. \tag{6.21}$$

The values of ϵ/β_* at which $\mathscr{C} = \mathscr{C}_{TP+}$ (so that all of S_+ except for points between the subcritical bifurcation and the turning point is stable) and $\hat{A}_+ = A_+$ (for which the subcritical bifurcation becomes supercritical and the turning point disappears) are plotted in figure 6. For pure capillary waves ($\sigma = 1$) all of R_1 is unstable for $\epsilon/\beta_* > 0.373$, all of the upper branch of S_+ is stable for $\epsilon/\beta_* > 0.750$, the turning point of R_1 disappears at $\epsilon/\beta_* = 1.55$, and the turning point of S_+ disappears at $\epsilon/\beta_* = 6.99$.

Turning to the coupled-mode solutions, we let $\dot{\mathscr{E}}_1 = \dot{\mathscr{E}}_2 = 0$ in (6.8*a*, *b*), and solve for

$$\mathscr{E} \equiv \mathscr{E}_1 + \mathscr{E}_2 = -(A+B+C)^{-1}(\mathscr{B}+\mathscr{C}) \quad (\mathscr{F}^2 > 0) \tag{6.22a}$$

and

$$\mathscr{F}^2 \equiv (\mathscr{E}_1 - \mathscr{E}_2)^2 = \frac{1-\mathscr{C}^2 + 2(A-B-C)\mathscr{C}\mathscr{D}\mathscr{E} - B^2\mathscr{E}^2}{(A-C)(A-2B-C)}. \tag{6.22b}$$

The straight line (6.22*a*), which corresponds to that of (5.1*c*) after the re-scaling (6.5) and the translation $\mathscr{C}$, terminates at the intersections with R_1 at $\mathscr{E} = \mathscr{F}$ and S_+ at $\mathscr{F} = 0$.

The characteristic equation for small perturbations about the solution (6.22) is

$$\lambda^2 + 2b\lambda + c = 0 \tag{6.23}$$

where

$$b = 2(A-C)^{-1}(A\hat{C} - \hat{A}C)\mathscr{E} - (A+C)B\mathscr{E}^2 + (A-C)(2A-B)\mathscr{F}^2 \tag{6.24a}$$

and

$$c = 16(A-C)^2(A+B+C)(2B+C-A)\mathscr{E}_1\mathscr{E}_2(\mathscr{E}_1-\mathscr{E}_2)^2. \tag{6.24b}$$

The coefficients b and c are both negative, and hence *CM* is unstable, for all admissible $\mathscr{E}$ if $kd \gg 1$ and $\sigma > \frac{1}{3}$.

The approximations (6.9), (6.13), (6.19) and (6.22*a*) are uniformly valid with respect to α and tend to (4.1*a*), (4.5*b*), (4.7*b*) and (5.1*a*), respectively, in the limit $\epsilon/\beta_* \downarrow 0$. But (6.12) and (6.24), which are based on the neglect of $\dot{\theta}_1$ and $\dot{\theta}_2$ in the evolution equations, are equivalent to their counterparts in §§4 and 5 only at the transition ($\lambda = 0$) points.

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Appendix A. Average Lagrangian

The quartic truncation of the Lagrangian for the motion described by (2.1) is (MH §2)

$$L = \tfrac{1}{2}(\delta_{mn} a_n + a_{lmn}\eta_l + \tfrac{1}{2}a_{jlmn}\eta_j\eta_l)\dot{\eta}_m\dot{\eta}_n - \tfrac{1}{2}[\delta_{mn}(g+\ddot{z}_0+\hat{T}k_n^2) - \tfrac{1}{4}\hat{T}b_{jlmn}\eta_j\eta_l]\eta_m\eta_n, \tag{A 1}$$

where $\dot{\eta}_n \equiv \mathrm{d}\eta_n/\mathrm{d}t$, g and $\ddot{z}_0$ are the gravitational and imposed accelerations, $\hat{T}$ is the kinematic surface tension, and $a_n \ldots$ are given by (MH (2.4)–(2.7))

$$a_1 = a_2 = (kT)^{-1}, \quad a_3 = a_4 = \tfrac{1}{4}(kT)^{-1}(1+T^2), \quad a_5 = \tfrac{1}{2}(kT)^{-1}S, \tag{A 2a–c}$$

$$T \equiv \tanh kd, \quad S \equiv \surd 2 \tanh kd/\tanh \surd 2kd, \tag{A 3a, b}$$

$$a_{113} = a_{131} = a_{224} = a_{242} = 2^{-\frac{3}{2}}(1-T^{-2}), \quad a_{311} = a_{422} = 2^{-\frac{1}{2}}(1+T^{-2}), \quad a_{512} = 1, \tag{A 4a–c}$$

$$b_{311} = b_{422} = 2^{-\frac{1}{2}}(3T^{-2}-1), \quad b_{512} = b_{521} = 2ST^{-2}-3, \tag{A 5a, b}$$

$$a_{1111} = a_{2222} = kT^{-3}, \quad a_{1122} = a_{2211} = k(ST^{-3}-2T^{-1}), \tag{A 6a, b}$$

$$a_{1221} = a_{2112} = kST^{-3}, \quad a_{1212} = a_{2121} = 0, \tag{A 6c, d}$$

$$b_{1111} = b_{2222} = \tfrac{3}{2}k^4, \quad b_{1122} = b_{2211} = k^4, \quad b_{1212} = b_{2121} = 0. \tag{A 7a c}$$

Proceeding as in Miles (1984) and MH §3, we pose the slowly varying amplitudes of the primary modes in the form

$$\eta_n = l[p_n(\tau)\cos\omega t + q_n(\tau)\sin\omega t] \quad (n = 1,2), \tag{A 8}$$

$$\tau = \epsilon\omega t, \tag{A 9}$$

where
$$l \equiv 2\epsilon^{\frac{1}{2}}k^{-1}\tanh kd, \quad \epsilon \equiv ka_0 \tanh kd. \tag{A 10a, b}$$

The amplitudes of the secondary modes, as determined from the requirement that the average Lagrangian be stationary with respect to their variations, then are given by (cf. Miles 1984, Appendix C)

$$\eta_n = (l^2/a_1)(A_n\cos 2\omega t + B_n \sin 2\omega t + C_n) \quad (n = 3,4,5), \tag{A 11}$$

where
$$(A_l, B_l) = \tfrac{1}{4}\Omega_l^{-1}\mathscr{b}_{lmn}(p_m p_n - q_m q_n, p_m q_n + p_n q_m), \tag{A 12a}$$

$$C_l = \tfrac{1}{4}\kappa_l^{-1}\, a_{lmn}(p_m p_n + q_m q_n), \tag{A 12b}$$

$$\Omega_n \equiv \frac{4k\tanh kd}{k_n \tanh k_n d} - \kappa_n, \quad \kappa_n \equiv \frac{1+k_n^2 l_*^2}{1+k^2 l_*^2}. \tag{A 13a, b}$$

The hypothesis that the primary modes dominate the secondary modes fails in the neighbourhood of $\Omega_3 = \Omega_4 = 0$ owing to the resonances between modes 1 and 3 ($k_3 = 2k_1$ and $\omega_3 = 2\omega_1$, corresponding to Wilton's ripples) and 2 and 4, which we exclude. Ω_5 is positive-definite.

Substituting z_0 and η_n from (1.1), (A 8) and (A 11) into (A 1), averaging L over a 2π interval of ωt, invoking (A 10) and (A 12), and evaluating the modal coefficients a_n, ..., for the ψ_n of (2.3) and (2.5), we obtain the average Lagrangian in the form

$$\langle L\rangle = a_0 l^2\omega^2[\tfrac{1}{2}(\dot{p}_n q_n - p_n \dot{q}_n) + H(p_n, q_n)], \tag{A 14}$$

where, here and subsequently, n is summed only over 1 and 2, the dots now imply differentiation with respect to the slow time τ,

$$H = \tfrac{1}{2}(p_n p_n - q_n q_n) + \tfrac{1}{2}\beta_n(p_n^2 + q_n^2) + \tfrac{1}{4}A_{mn}(p_m^2 + q_m^2)(p_n^2 + q_n^2) + \tfrac{1}{2}B(p_1 q_2 - p_2 q_1)^2 + \tfrac{1}{2}P_{mn}(p_m^2 p_n^2 - q_m^2 q_n^2) \tag{A 15}$$

is a Hamiltonian for the slowly varying amplitudes,

$$\beta_1 = \beta_2 = \beta = (\omega - \omega_k)/\epsilon\omega \tag{A 16}$$

is a measure of the frequency offset from (linear) resonance,

$$A_{11} = A_{22} \equiv A = \tfrac{1}{2} + \tfrac{9}{8}T^2\sigma + \frac{1}{4}\frac{(1+T^2)^2}{\kappa_3} - \frac{1}{8}\frac{(3-T^2)^2}{\Omega_3}, \tag{A 17a}$$

$$A_{12} = A_{21} \equiv C = S - T^2 + \tfrac{3}{4}T^2\sigma + \frac{T^4}{\kappa_5} - \frac{1}{2}\frac{(2S-3T^2)^2}{\Omega_5}, \quad B = -2T^2 - \tfrac{1}{4}T^2\sigma - \frac{T^4}{\kappa_5}, \tag{A 17b, c}$$

$$P_{11} = P_{22} \equiv P = \tfrac{1}{2}\epsilon\frac{(1+T^2)(3-T^2)}{\kappa_3\Omega_3}, \quad P_{12} = P_{21} \equiv Q = 2\frac{\epsilon T^2(2S-3T^2)}{\kappa_5\Omega_5}, \tag{A 18a, b}$$

$$\sigma \equiv k^2 l_*^2/(1+k^2 l_*^2). \tag{A 19}$$

We remark that (A 14)–(A 18) are invariant under the interchange of the subscripts 1 and 2. The deep-water limits ($kd \uparrow \infty$) of (A 17) and (A 18) are given by (3.5) and (6.3).

The requirement (Hamilton's principle) that $\langle L\rangle$ be stationary with respect to independent variations of p_n and q_n implies the canonical equations

$$\dot{p}_n = -\partial H/\partial q_n, \quad \dot{q}_n = \partial H/\partial p_n, \qquad \text{(A 20}a, b\text{)}$$

from which (3.2) are derived through the introduction of dissipation.

Appendix B. Dissipation ($kd \gg 1$)

We hypothesize the dissipation function†

$$D = \alpha(e_1+e_2)+\mathscr{D}+O(\epsilon^5), \quad \mathscr{D} = \tilde{A}_{mn}e_m e_n \qquad \text{(B 1}a, b\text{)}$$

on the assumptions that $\tilde{A}_{mn}$, e_1, e_2 and $\theta_1-\theta_2$ are $O(\epsilon)$, as in §6. It suffices for the determination of the $\tilde{A}_{mn}$ to consider the decay of free oscillations, for which $\omega \equiv \omega_1$ (but note that the modulated frequency of η_n in (3.1) is $\omega-\epsilon\omega\dot{\theta}_n$) and (3.3$b$) reduces to

$$H = A_{mn}e_m e_n + \tfrac{1}{2}Br^2. \qquad \text{(B 2)}$$

Substituting (B 1) and (B 2) into (3.11) and invoking $e \equiv e_1+e_2$, we obtain

$$\dot{e} = \dot{e}_1+\dot{e}_2 = -2\alpha e - 4\mathscr{D}+O(\epsilon^5), \qquad \text{(B 3}a\text{)}$$

$$\dot{\theta}_1 = 2Ae_1+2Ce_2+O(\epsilon^3), \quad \dot{\theta}_2 = 2Ae_2+2Ce_1+O(\epsilon^3). \qquad \text{(B 3}b, c\text{)}$$

We construct an alternative expression for $\dot{e}$ through the mean energy equation,

$$(\mathrm{d}/\mathrm{d}t)\langle E\rangle = -2\langle F\rangle, \qquad \text{(B 4)}$$

in which ρE and ρF are the energy and Rayleigh dissipation function per unit surface area for the free oscillations. Posing

$$\langle E\rangle = k^{-1}l^2\omega^2(e+\mathscr{E}), \quad \langle F\rangle = \alpha\epsilon k^{-1}l^2\omega^3(e+\mathscr{F}), \qquad \text{(B 5}a, b\text{)}$$

wherein (by hypothesis) $\mathscr{E}$ and $\mathscr{F}$ are quadratic in e_1 and e_2, and invoking

$$(\mathrm{d}/\mathrm{d}t)(e+\mathscr{E}) = \epsilon\omega(\dot{e}-4\alpha\mathscr{E}), \qquad \text{(B 6)}$$

where $\mathrm{d}\mathscr{E}/\mathrm{d}\tau = \dot{\mathscr{E}} = -4\alpha\mathscr{E}$ follows from the first approximation to (B 3a), $\dot{e} = -2\alpha e$, we obtain

$$\dot{e} = -2\alpha(e+\mathscr{F})+4\alpha\mathscr{E}. \qquad \text{(B 7)}$$

Comparison with (B 3a) then yields

$$\mathscr{D} = \alpha(\tfrac{1}{2}\mathscr{F}-\mathscr{E}). \qquad \text{(B 8)}$$

Proceeding as in Miles (1976) and invoking $kd \gg 1$ and the expansion (cf. (2.1))

$$\phi = \phi_n(t)\psi_n(\boldsymbol{x})\,\mathrm{e}^{k_n z} \qquad \text{(B 9)}$$

(which follows from $\nabla^2\phi = 0$ and $kd \gg 1$) for the velocity potential, we obtain

$$E = \tfrac{1}{2}\mathscr{k}_{mn}\phi_m\phi_n + \tfrac{1}{2}[\delta_{mn}(g+Tk_n^2)-\tfrac{1}{4}T\mathscr{b}_{jlmn}\eta_j\eta_l]\eta_m\eta_n, \qquad \text{(B 10)}$$

where

$$\mathscr{k}_{mn} = \delta_{mn}k_n + (C_{lmn}k_m k_n + D_{lmn})\eta_l + \tfrac{1}{2}(C_{jlmn}k_m k_n + D_{jlmn})(k_m+k_n)\eta_j\eta_l. \qquad \text{(B 11)}$$

The ϕ_n, calculated as in §2 of Miles (1976) with $kd \gg 1$, are given by

$$\phi_1 = k^{-1}\dot{\eta}_1 + 2^{-\frac{1}{2}}(\eta_3\dot{\eta}_1-\eta_1\dot{\eta}_3-\eta_2\dot{\eta}_5)+k[\tfrac{3}{4}\eta_1^2\dot{\eta}_1+\tfrac{1}{2}(\sqrt{2}-1)\eta_2^2\dot{\eta}_1+2^{-\frac{1}{2}}\eta_1\eta_2\dot{\eta}_2], \qquad \text{(B 12}a\text{)}$$

† The symbols $\mathscr{D}$, $\mathscr{E}$ and $\mathscr{F}$ now have different definitions *vis-à-vis* §6.

$$\phi_3 = \tfrac{1}{2}k^{-1}\dot{\eta}_3 - 2^{-\frac{1}{2}}\eta_1\dot{\eta}_1, \quad \phi_5 = 2^{-\frac{1}{2}}(k^{-1}\dot{\eta}_5 - \eta_1\dot{\eta}_2 - \eta_2\dot{\eta}_1), \qquad (\text{B } 12b, c)$$

and ϕ_2 and ϕ_4 by (B 12a, b) with the subscripts (1,2,3) replaced by (2,1,4), respectively.

The Rayleigh dissipation function, on the assumptions of an uncontaminated free surface and $kd \gg 1$, may be approximated by (Lamb 1932, §329(7), divided by ρS)

$$F = \nu S^{-1} \iint \mathrm{d}S \int_{-\infty}^{\eta} (\phi_{xx}^2 + \phi_{yy}^2 + \phi_{zz}^2 + 2\phi_{xy}^2 + 2\phi_{yz}^2 + 2\phi_{xz}^2)\,\mathrm{d}z, \qquad (\text{B } 13)$$

where ν is the kinematic viscosity. Invoking (B 9) and truncating at fourth order, we obtain

$$F = 2\nu[k_m k_n \ell_{mn} + E_{lmn}\eta_l + E_{jlmn}\eta_j\eta_l]\phi_m\phi_n, \qquad (\text{B } 14)$$

where

$$E_{lmn} = \langle \psi_l \chi_{mn} \rangle, \quad E_{jlmn} = \tfrac{1}{2}(k_m + k_n)\langle \psi_j \psi_l \chi_{mn} \rangle, \quad \chi_{mn} \equiv \psi_{mxy}\psi_{nxy} - \psi_{mxx}\psi_{nyy}. \qquad (\text{B } 15a\text{–}b)$$

The dominant components of the means of (B 10) and (B 14) are

$$\langle E \rangle_{\text{dom}} = \tfrac{1}{2}k\langle \phi_1^2 + \phi_2^2 \rangle + \tfrac{1}{2}(g + Tk^2)\langle \eta_1^2 + \eta_2^2 \rangle = \tfrac{1}{2}k^{-1}l^2(\omega^2 + \omega_1^2)\,e \qquad (\text{B } 16a)$$

and

$$\langle F \rangle_{\text{dom}} = 2\nu k^3 \langle \phi_1^2 + \phi_2^2 \rangle = 2\nu k^2(\omega^2 l^2/k)\,e, \qquad (\text{B } 16b)$$

which, after invoking $\omega^2 = \omega_1^2$ and $\delta = 2\nu k^2/\omega$ (linear Stokes damping), correspond to the anticipated terms (in e) in (B 5a, b). It then follows from (B 5), (B 8), and $g + Tk_n^2 = \kappa_n \omega^2/k$, where κ_n is given by (A 13b), that

$$\mathscr{D} = \tfrac{1}{2}\alpha k^{-1}(l\omega)^{-2}\langle [(k_m k_n - k^2)\ell_{mn} + E_{lmn}\eta_l + E_{jlmn}\eta_j\eta_l]\phi_m\phi_n - (1 - \delta_{1n} - \delta_{2n})k\omega^2\kappa_n\eta_n^2 + \tfrac{1}{4}Tk^2 \mathscr{C}_{jlmn}\eta_j\eta_l\eta_m\eta_n \rangle. \qquad (\text{B } 17)$$

Substituting (B 12) into (B 17), we obtain (after a straightforward but non-trivial reduction)

$$\begin{aligned}\mathscr{D} = \tfrac{1}{2}\alpha(l\omega)^{-2}\Big\langle \Big(\frac{k_n}{k} - \frac{k}{k_n}\Big)\dot{\eta}_n^2 &- (1 - \delta_{n1} - \delta_{n2})\,\omega^2\kappa_n\eta_n^2 \\ &- k[\surd 2(\eta_1\dot{\eta}_1\dot{\eta}_3 + \eta_2\dot{\eta}_2\dot{\eta}_4) + \eta_5\dot{\eta}_1\dot{\eta}_2 + \tfrac{1}{2}\surd 2(\eta_1\dot{\eta}_2 + \eta_2\dot{\eta}_1)\,\dot{\eta}_5] \\ &- k^2(\eta_1\dot{\eta}_1 + \eta_2\dot{\eta}_2)^2 + \tfrac{1}{4}Tk^5[\tfrac{3}{2}(\eta_1^4 + \eta_2^4) + 2\eta_1^2\eta_2^2]\Big\rangle. \end{aligned} \qquad (\text{B } 18)$$

The amplitudes of the secondary modes, as determined from the requirement that the average Lagrangian be stationary with respect to their variations, are given by (cf. MH, Appendix C)

$$\eta_n = (l^2/a_1)(A_n \cos 2\omega t + B_n \sin 2\omega t + C_n) \quad (n = 3, 4, 5), \qquad (\text{B } 19)$$

where

$$(A_l, B_l) = \tfrac{1}{4}\Omega_l^{-1}\mathscr{C}_{lmn}(p_m p_n - q_m q_n, p_m q_n + p_n q_m) \qquad (\text{B } 20a)$$

and

$$C_l = \tfrac{1}{4}\kappa_l^{-1} a_{lmn}(p_m p_n + q_m q_n). \qquad (\text{B } 20b)$$

Combining (3.1) and (B 20) in (B 18), averaging over ωt, invoking $kd \gg 1$, and comparing the result for $\mathscr{D}$ with the quartic terms in (B 3), we obtain

$$\tilde{A}_{11} = \tilde{A}_{22} \equiv \tilde{A} = \alpha\epsilon(-1 - \kappa_3^{-1} - \tfrac{3}{2}\Omega_3^{-1} + 2\Omega_3^{-2} + \tfrac{9}{8}\sigma), \qquad (\text{B } 21a)$$

$$\tilde{A}_{12} \equiv \tilde{C} = \alpha\epsilon[-1 - 2\kappa_5^{-1} + (3 - 2\surd 2)\,\Omega_5^{-1} + \tfrac{3}{4}\sigma], \quad \tilde{B} = \alpha\epsilon(2 + 2\kappa_5^{-1} - \tfrac{1}{2}\sigma). \qquad (\text{B } 21b, c)$$

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